Second order brane cosmology with radion stabilization

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We study cosmology in the five-dimensional Randall-Sundrum brane-world with a stabilizing effective potential for the radion and matter localized on the branes. The analysis is performed by employing a perturbative expansion in the ratio ρ/V between the matter energy density on the branes and the brane tensions around the static Randall-Sundrum solution (which has $\rho = 0$ and brane tensions $\pm V$). This approach ensures that the matter evolves adiabatically and allows us to find approximate solutions to second order in ρ/V. Some particular cases are then analyzed in details.

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I. INTRODUCTION

Higher and higher precision data which are about to be collected in new experiments of particle physics and astrophysics in the next few years convey considerable attention to theories with extra dimensions. The main role of such theories, originally introduced in the 20's by Kaluza and Klein [1, 2], is to provide a connection between particle physics and gravity at some level. At a deeper level, string theory unifies all the interactions by means of some n-dimensional manifold (with $n > 4$) where the fundamental objects are supposedly living; at a more phenomenological level, models which assume the existence of extra dimensions, no matter their origin, are considered in order to solve some puzzles of particle physics, cosmology and astrophysics, giving rise to many possible observable consequences.

Originally proposed in order to solve the problem of the large hierarchy between Gravity and Standard Model scales, the Randall-Sundrum model of Ref. [3] (RS I) has acquired considerable relevance due to its stringy inspiration. It represents the prototype of the so-called brane-world and differs from previous models in that it constrains standard matter on a four-dimensional manifold (the brane) just letting gravity (and exotic matter) propagate everywhere. The RS I solution to the hierarchy problem needs one additional compactified (orbifolded) spatial dimension with two branes located at its fixed points, plus a negative cosmological constant filling the space between such branes (the bulk). The bulk cosmological constant Λ warps the extra dimension and generates the effective four-dimensional physical constants we measure. It was soon realized that the modifications to four-dimensional gravity induced by the fifth dimension may be reduced to such a short distance effect to be unobservable even in the presence of just one brane and infinite compactification radius (the RS II model of Ref. [4]).

The cosmological features of the RS models are nowadays being investigated even more than its particle physics consequences, due to the refined results lately obtained and to the major problems recent astrophysical data have revealed: the possible late time acceleration from supernovae, CMBR spectrum, dark matter and dark energy quests suggest either a full revision of the modern theoretical physics approach or the possibility of the existence of further, up to now ignored, ingredients such as the extra dimensions.

In particular the single brane RS II cosmological dynamics [5, 6] is known to generate $(\rho/V)^2$ corrections to standard Friedmann and acceleration equations, where ρ is the energy density of the fluid filling the brane and V is the constant brane tension. These corrections are negligible when $\rho \ll V$, the regime in which the RS II model is reliable and leads to standard cosmic evolution. The two brane RS I setup is much more involved: a stabilization mechanism for the distance between the branes, such as that of Ref. [7], is necessary to get the correct hierarchy in the absence of matter. Moreover, a bulk potential for the radion (the metric degree of freedom associated with the fifth dimension) is necessary to achieve solvable junction conditions when matter is present on the boundaries [8]. In this case, cosmological solutions to order ρ/V [9] are not sufficient to grasp the particular features of the background metric evolution originated by the extra dimension and one needs to investigate the effect of terms of order $(\rho/V)^2$ (as was done in Ref. [10]) or higher.

The aim of this article is to go beyond the first order approximation in brane cosmology for RS I models with two branes. Our approach will differ from Ref. [10] in that we do not consider a bulk scalar field to stabilize the radion but include an effective stabilizing potential directly into the equations (see also Ref. [11]). Consequently, our perturbative expansion is around the RS I solution. The calculations are then carried out in order to show how ρ^2 contributions to the four-dimensional

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Hubble parameter may affect the model (or may be unobservable). Such terms are expected as fingerprints of the fifth dimension in analogy with the single brane RS II framework. The latter case will also be studied as RS I in the limit when the distance between the branes diverges. Some hints about the possibility of an accelerated expansion driven by exotic fluids with pressure $p = w\,\rho$ and $w > 0$ will be presented, thus suggesting the necessity to go beyond the second order approximation.

The paper is organized as follows: in Section II, we present the complete setup of the model under consideration; in Section III the second order ansatz is described and Einstein equations are perturbatively solved; in Section IV cosmological consequences of the solutions are analyzed and compared to the known brane-world solutions; in Section V the analysis of the approximations is performed and, finally, in Section VI, some conclusions are drawn. For the five-dimensional metric g_{AB} we shall use the signature $(+,-,-,-,-)$, so that $g \equiv \det(g_{AB}) > 0$.

II. EINSTEIN EQUATIONS

Let us consider a RS I model perturbed by the presence of matter on the two branes. The bulk metric is given by

$$\begin{aligned} \mathrm{d}s^2 &\equiv g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B \\ &= n^2(y,t)\,\mathrm{d}t^2 - a^2(y,t)\,\mathrm{d}x^i\,\mathrm{d}x^i - b^2(y,t)\,\mathrm{d}y^2 \ . \end{aligned} \tag{1}$$

The Einstein tensor for this metric is

$$G_{00}=3\left\{\left(\frac{\dot a}{a}\right)^2+\frac{\dot a\,\dot b}{a\,b}-\frac{n^2}{b^2}\left[\frac{a''}{a}+\left(\frac{a'}{a}\right)^2-\frac{a'\,b'}{a\,b}\right]\right\} \tag{2a}$$

$$\begin{aligned} G_{ii}=&\frac{a^2}{b^2}\left[\left(\frac{a'}{a}\right)^2+2\frac{a'\,n'}{a\,n}-\frac{b'\,n'}{b\,n}-2\frac{a'\,b'}{a\,b}+2\frac{a''}{a}+\frac{n''}{n}\right] \\ &-\frac{a^2}{n^2}\left[\left(\frac{\dot a}{a}\right)^2-2\frac{\dot a\,\dot n}{a\,n}+2\frac{\ddot a}{a}-\frac{\dot b}{b}\left(\frac{\dot n}{n}-2\frac{\dot a}{a}\right)+\frac{\ddot b}{b}\right] \end{aligned} \tag{2b}$$

$$G_{04}=3\left[\frac{\dot a\,n'}{a\,n}+\frac{a'\,\dot b}{a\,b}-\frac{\dot a'}{a}\right] \tag{2c}$$

$$G_{44}=3\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right)-\frac{b^2}{n^2}\left[\frac{\dot a}{a}\left(\frac{\dot a}{a}-\frac{\dot n}{n}\right)+\frac{\ddot a}{a}\right]\right\} \tag{2d}$$

where a prime denotes a derivative with respect to y and a dot a derivative with respect to the universal time t. The energy-momentum tensor in the bulk is that of an anti-de Sitter space with the addition of a term generated by a field which serves the purpose of stabilizing the distance between the two branes of the RS I model,

$$T^A_{\ B} = \Lambda\, g^A_{\ B} + \tilde T^A_{\ B} \ , \tag{3}$$

where, as usual,

$$\tilde T_{AB} = -\frac{2}{\sqrt{g}}\,\frac{\delta\mathcal{L}_{\rm stab}}{\delta g^{AB}} \ , \tag{4}$$

and $\mathcal{L}_{\rm stab}$ is the Lagrangian of the stabilizing field (for a scalar field, see e.g. Ref. [7, 9, 10]).

We shall consider the particular case in which the stabilizing mechanism can be effectively described by a harmonic potential for the radion, with an effective Lagrangian of the form [8]

$$\mathcal{L}_{\rm eff} = -\sqrt{g}\,\omega^2\,(b-b_0)^2 \equiv -\sqrt{g}\,U(b) \ , \tag{5}$$

when the metric is written as in Eq. (1), and the potential U depends on the component g_{44}. The two 3-branes have opposite tensions and their contribution to the total energy-momentum tensor is given by

$$\begin{aligned} T^A_{i\,B} = &\frac{\delta(y-y_i)}{b} \\ &\times \mathrm{diag}\,(V_i+\rho_i, V_i-p_i, V_i-p_i, V_i-p_i, 0) \ , \end{aligned} \tag{6}$$

where $i = p,\ n$, and $y_p = 0$ ($y_n = 1/2$) is the position of the positive (negative) tension brane. The Einstein equations in the bulk,

$$G_{AB} = k^2\,T_{AB} \ , \tag{7}$$

form a system of four differential equations for the three independent functions $f_\alpha = (n,a,b)$. On using the Bianchi identity $\nabla_A G^{A0} = 0$, it is then straightforward to show that the three equations

$$\begin{cases} G_{00} = k^2\,T_{00} \\ G_{04} = 0 \\ G_{44} = k^2\,T_{44} \end{cases} \tag{8}$$

are independent and form a complete set. This means that a solution to Eqs. (8) also solves the full set (7). Moreover, since G_{AB} is trivially conserved because of the Bianchi identities, the tensor T_{AB} is also automatically conserved regardless of the fact that the effective potential in Eq. (5) does not appear covariant.

For computational purposes, it is convenient to do some further manipulation. The first of Eqs. (8) can be replaced by [5, 6]

$$F'(y,t) + \frac{1}{6}\,k^2\,\left(\frac{\partial}{\partial y}a^4\right)\,T^0_{\ 0} = 0 \ , \tag{9}$$

with

$$F(y,t) = \frac{(a\,a')^2}{b^2} - \frac{(a\,\dot a)^2}{n^2} \ , \tag{10}$$

which, on intergating along the extra dimension, can be written as

$$\begin{aligned} &\frac{\dot a^2}{a^2} - \frac{n^2\,a'^2}{b^2\,a^2} - \frac{k^2}{6}n^2\,T^0_{\ 0} + \frac{k^2\,n^2}{6\,a^4}\int a^4\,\left(T^0_{\ 0}\right)'\,\mathrm{d}y \\ &= \frac{n^2}{a^4}\,\tilde c(t) \ , \end{aligned} \tag{11}$$

where $\tilde{c}(t)$ is related to the boundary conditions at the branes. The conservation equation $\nabla_A T^{A4} = 0$ yields

$$b' = (b_0 - b)\left(\frac{n'}{n} + 3\frac{a'}{a} + 2\frac{b'}{b}\right) , \tag{12}$$

which is identically satisfied by the solutions of the system (8). Instead of the three Eqs. (8), we shall therefore solve the equivalent system

$$\begin{cases} \frac{\dot{a}^2}{a^2} - \frac{n^2 a'^2}{b^2 a^2} - \frac{k^2}{6} n^2 T^0_{\ 0} + \frac{k^2 n^2}{6\, a^4} \int a^4 \left(T^0_{\ 0}\right)' \mathrm{d}y = \frac{n^2}{a^4}\tilde{c}(t) \\ b' = (b_0 - b)\left(\frac{n'}{n} + 3\frac{a'}{a} + 2\frac{b'}{b}\right) \\ G_{44} = k^2\, T_{44} \ . \end{cases} \tag{13}$$

Moreover, bulk solutions must satisfy the boundary equations given by the junction conditions on the two branes,

$$\begin{cases} \lim\limits_{y\to y_i^+} \dfrac{a'}{a} = -\dfrac{k^2}{6}\left(V_i + \rho_i\right)\, b\Big|_{y=y_i} \\ \lim\limits_{y\to y_i^+} \dfrac{n'}{n} = -\dfrac{k^2}{6}\left[V_i - (2 + 3\, w_i)\,\rho_i\right]\, b\Big|_{y=y_i} \end{cases} , \tag{14}$$

where we assumed an equation of state for the vacuum perturbations of the form $p_i = w_i\,\rho_i$. When $\rho_i \to 0$ the RS I solution is fully recovered and one finds the usual warped static metric with $\tilde{c}(t) = 0$ and

$$\begin{aligned} & n_{RS}(y,t) = a_{RS}(y,t) = \exp\left(-m\, b_0\, |y|\right) \\ & b_{RS} = b_0 \ , \end{aligned} \tag{15}$$

which also require the well known fine-tuning

$$V_p = -V_n = \frac{6m}{k^2} \ , \qquad \Lambda = -\frac{6m^2}{k^2} \ . \tag{16}$$

III. THE LOW DENSITY EXPANSION

A perturbative approach can be adopted in brane cosmology to investigate solutions to the Einstein equations by taking as a starting point the static RS I metric with $\rho_i = 0$ reviewed in the previous section. In fact, in the low density regime

$$\frac{\rho_i}{|V_i|} \ll 1 \ , \tag{17}$$

one can express the corrections to the solution (15) to all orders in ρ_i/V_i by assuming that the metric functions $f_\alpha = (n, a, b)$ can be written as [12]

$$f_\alpha = f_{RS} + \delta f_\alpha \ , \tag{18}$$

with $\delta f_\alpha \sim \sum_{n_i, n_j \geq 1} c_{n_i n_j}\, \rho_i^{n_i}\, \rho_j^{n_j}$.

In order to keep track of the various orders in the above expansion, it is useful to introduce an expansion parameter ϵ by replacing $\rho_i \to \epsilon\,\rho_i$ (and setting $\epsilon = 1$ at the end of the computation). We make the following *ansatz* for the metric,

$$n(y,t) = \exp\left(-m\, b_0\, |y|\right)\,\left[1 + \delta f_n(y,t)\right] \tag{19a}$$

$$a(y,t) = a_h(t)\,\exp\left(-m\, b_0\, |y|\right)\,\left[1 + \delta f_a(y,t)\right] \tag{19b}$$

$$b(y,t) = b_0 + \delta f_b(y,t) \ , \tag{19c}$$

so that the homogeneous scale factor $a_h(t)$ is factored out, and b_0 is the equilibrium point corresponding to the RS I model [see Eq. (15) above]. The solutions to the equations (13) can then be completely expressed in terms of the functions δf_α and $H_h \equiv \dot{a}_h/a_h$, which we expand to second order in ϵ as

$$\begin{aligned} \delta f_\alpha \ &\simeq\ \epsilon\,\left[f^{(1)}_{\alpha,p}(y)\,\rho_p + f^{(1)}_{\alpha,n}(y)\,\rho_n\right] \\ &\quad + \epsilon^2\,\left[f^{(2)}_{\alpha,p}(y)\,\rho_p^2 + f^{(2)}_{\alpha,n}(y)\,\rho_n^2 + f^{(2)}_{\alpha,m}(y)\,\rho_p\,\rho_n\right] \end{aligned} \tag{20a}$$

$$\begin{aligned} H_h^2 \ &\simeq\ \epsilon\,\left(h^{(1)}_{h,p}\,\rho_p + h^{(1)}_{h,n}\,\rho_n\right) \\ &\quad + \epsilon^2\,\left(h^{(2)}_{h,p}\,\rho_p^2 + h^{(2)}_{h,n}\,\rho_n^2 + h^{(2)}_{h,m}\,\rho_p\,\rho_n\right) \ . \end{aligned} \tag{20b}$$

We also expand $c \equiv \tilde{c}/a_h^4$ as

$$\begin{aligned} c(t) \ &\simeq\ \epsilon\,\left(c^{(1)}_p\,\rho_p + c^{(1)}_n\,\rho_n\right) \\ &\quad + \epsilon^2\,\left(c^{(2)}_p\,\rho_p^2 + c^{(2)}_n\,\rho_n^2 + c^{(2)}_m\,\rho_p\,\rho_n\right) \ . \end{aligned} \tag{20c}$$

Note that the time dependence is just carried by the functions $\rho_i = \rho_i(t)$ and that, of all the coefficients appearing above, only the $f^{(n)}_{\alpha,i}$'s in Eq. (20a) depend on y, whereas the others are constant.

In order to proceed with the perturbative expansion, one also needs to expand $\dot{\rho}_i$. From the conservation equation [13]

$$\dot{\rho}_i = -3\, H(y_i, t)\,\left(1 + w_i\right)\,\rho_i \ , \tag{21}$$

it immediately follows that the time evolution of the matter densities is adiabatic, since $H \equiv \dot{a}/a \sim \rho^{1/2}$ and, therefore,

$$\frac{|\dot{\rho}_i|}{\rho_i^{5/4}} \sim \left(\frac{\rho_i}{|V_i|}\right)^{1/4} \ll 1 \ . \tag{22}$$

If we now assume that, to second order in ϵ,

$$\begin{aligned} H^2(y,t) \simeq\ &\epsilon\,\left[h^{(1)}_p(y)\,\rho_p + h^{(1)}_n(y)\,\rho_n\right] \\ &+ \epsilon^2\,\left[h^{(2)}_p(y)\,\rho_p^2 + h^{(2)}_n(y)\,\rho_n^2 + h^{(2)}_m(y)\,\rho_p\,\rho_n\right] \ , \end{aligned} \tag{23}$$

the coefficients of the above expansion can be related to the corresponding ones in Eqs. (20b) and (20a) for $\alpha = a$

by equating the two expressions for H^2 at $y=y_i$ up to second order,

$$H^2(y_i,t) = \left(H_h(t) + \frac{\delta\dot{f}_a(y_i,t)}{1+\delta f_a(y_i,t)}\right)^2 \ . \tag{24}$$

With the help of Eq. (21), one finally obtains

$$h_i^{(1)}(y) \ = \ h_{h,i}^{(1)} \tag{25a}$$

$$h_i^{(2)}(y) \ = \ h_{h,i}^{(2)} - 6(1+w_i)\, h_{h,i}^{(1)}\, f_{a,i}^{(1)}(y) \tag{25b}$$

$$\begin{aligned} h_m^{(2)}(y) \ = \ & h_{h,m}^{(2)} - 6(1+w_p)\, h_{h,n}^{(1)}\, f_{a,p}^{(1)}(y) \\ & -6(1+w_n)\, h_{h,p}^{(1)}\, f_{a,n}^{(1)}(y) \ . \end{aligned} \tag{25c}$$

A. First order results

In order to solve the bulk equations order by order, one has to substitute the previous expansion in the dynamical equations (13). This will allow us to determine explicitly the coefficients δf_α once the boundary conditions are imposed. Let us begin with first order equations.

At order ϵ, the constraint (12) reads

$$f_{b,p}^{(1)\prime}\rho_p + f_{b,n}^{(1)\prime}\rho_n - 4\, m\, b_0 \left[f_{b,p}^{(1)}\,\rho_p + f_{b,n}^{(1)}\,\rho_n\right] = 0 \ . \tag{26}$$

If we allow ρ_p and ρ_n to be arbitrary functions of the time, the above equation splits into two independent equations for the coefficients of ρ_i, and one finds

$$f_{b,i}^{(1)}(y) = b_i^{(1)} \, \exp\left(4\, m\, b_0\, y\right) \ , \tag{27}$$

where $b_i^{(1)}$ are constant coefficients to be determined.

The functions $f_{a,i}^{(1)}(y)$'s can now be calculated by solving Eq. (11). Since the contribution of the stabilizing potential vanishes at order ϵ, the integral-differential equation (11) becomes the first order linear differential equation

$$\sum_i \mathrm{e}^{-2\, m\, b_0\, y}\, \rho_i \left(2\, m^2\, b_i^{(1)}\, \mathrm{e}^{4\, m\, b_0\, y} - b_0\, c_i^{(1)}\, \mathrm{e}^{4\, m\, b_0\, y} + b_0\, h_{h,i}^{(1)}\, \mathrm{e}^{2\, m\, b_0\, y} + 2\, m\, f_{a,i}^{(1)\prime}\right) = 0 \ . \tag{28}$$

On solving for each coefficient of ρ_i independently, one obtains

$$f_{a,i}^{(1)} \ = \ \frac{1}{8}\,\mathrm{e}^{2\, m\, b_0\, y}\left[\mathrm{e}^{2\, m\, b_0\, y}\left(\frac{c_i^{(1)}}{m^2} - 2\,\frac{b_i^{(1)}}{b_0}\right) - 2\,\frac{h_{h,i}^{(1)}}{m^2}\right] + c_{a,i}^{(1)} \ . \tag{29}$$

Finally, one can expand the equation $G_{44} = k^2\, T_{44}$ to first order,

$$\sum_i \rho_i \left[\mathrm{e}^{2\, m\, b_0\, y}\left(6b_0 h_{h,i}^{(1)} + 9b_0 w_i h_{h,i}^{(1)} - 9b_0 c_i^{(1)}\mathrm{e}^{2\, m\, b_0\, y} - 6m^2 b_i^{(1)}\mathrm{e}^{2\, m\, b_0\, y} + 4k^2\omega^2 b_0^2 b_i^{(1)}\mathrm{e}^{2\, m\, b_0\, y}\right) - 6m f_{n,i}^{(1)}\right] = 0 \ , \tag{30}$$

and solve the two equations for $f_{n,i}^{(1)}(y)$. The result is

$$f_{n,i}^{(1)} = \frac{1}{2}\,\mathrm{e}^{2\, m\, b_0\, y}\left[\mathrm{e}^{2\, m\, b_0\, y}\left(\frac{k^2\,\omega^2}{3\, m^2}\, b_0\, b_i^{(1)} - \frac{3\, c_i^{(1)}}{4\, m^2} - \frac{b_i^{(1)}}{2\, b_0} + \frac{(2+3\, w_i)}{2\, m^2}\, h_{h,i}^{(1)}\right)\right] + c_{n,i}^{(1)} \ . \tag{31}$$

We are now left with six numerical coefficients

$$h_{h,i}^{(1)} \ , \quad b_i^{(1)} \ , \quad c_i^{(1)} \ , \tag{32a}$$

and four integration constants

$$c_{a,i}^{(1)} \ , \quad c_{n,i}^{(1)} \ . \tag{32b}$$

In order to fix the above, one has to use the junction conditions. These four conditions, written in terms of the coefficients of ρ_i, form a system of eight equations: the discontinuity constraints for $a'(y,t)$ at $y=y_i$ imply

$$\left\{\begin{aligned} & \rho_p\left(3\, c_p^{(1)} - 3\, h_{h,p}^{(1)} + k^2\, m\right) + 3\,\rho_n\left(c_n^{(1)} - h_{h,n}^{(1)}\right) = 0 \\ & 3\,\rho_p\left(c_p^{(1)}\mathrm{e}^{m\, b_0} - h_{h,p}^{(1)}\right) \\ & +\rho_n\left(3\, c_n^{(1)}\mathrm{e}^{2\, m\, b_0} - 3\, h_{h,n}^{(1)}\mathrm{e}^{m\, b_0} - k^2\, m\right) = 0 \ , \end{aligned}\right. \tag{33}$$

whose solution is given by

$$h^{(1)}_{h,p} = \frac{k^2\, m\, \mathrm{e}^{m\, b_0}}{3\, (\mathrm{e}^{m\, b_0} - 1)} \;, \qquad h^{(1)}_{h,n} = \frac{k^2\, m\, \mathrm{e}^{-m\, b_0}}{3\, (\mathrm{e}^{m\, b_0} - 1)} \quad \text{(34a)}$$

$$c^{(1)}_{p} = \frac{k^2\, m}{3\, (\mathrm{e}^{m\, b_0} - 1)} \;, \qquad c^{(1)}_{n} = \frac{k^2\, m\, \mathrm{e}^{-m\, b_0}}{3\, (\mathrm{e}^{m\, b_0} - 1)} \;. \quad \text{(34b)}$$

The two analogous constraints for $n'(y,t)$ are both equivalent to the equation

$$\begin{aligned} &\mathrm{e}^{m\, b_0} \left[m\, (3\, w_p - 1) + 4\, \omega^2\, b_0\, b^{(1)}_p \left(\mathrm{e}^{m\, b_0} - 1 \right) \right] \rho_p \qquad (35) \\ &+ \left[m\, (3\, w_n - 1) + 4\, \omega^2\, b_0\, b^{(1)}_n \mathrm{e}^{m\, b_0} \left(\mathrm{e}^{m\, b_0} - 1 \right) \right] \rho_n = 0 \;, \end{aligned}$$

which yields

$$\begin{aligned} b^{(1)}_p &= \frac{m\, (3\, w_p - 1)}{4\, \omega^2\, b_0\, (\mathrm{e}^{m\, b_0} - 1)} \\ b^{(1)}_n &= \frac{m\, (3\, w_n - 1)\, \mathrm{e}^{-m\, b_0}}{4\, \omega^2\, b_0\, (\mathrm{e}^{m\, b_0} - 1)} \;. \end{aligned} \qquad (36)$$

We see that the junction conditions are not sufficient to determine the integration constants (32b). Such freedom is in fact related to the gauge freedom in the choice of the initial value for the scale factor and time variable. Without loss of generality, and to simplify the second order calculations, we then set $c^{(1)}_{a,i} = 0$. The values of the $c^{(1)}_{n,i}$'s are related to the choice of the time variable. Since one usually considers the negative tension brane in RS I as the four-dimensional "visible Universe", it is natural to use the proper time τ on this brane and choose the $c^{(1)}_{n,i}$'s so as to have $n(y_n, \tau) = 1$. This can be achieved by setting

$$\begin{aligned} c^{(1)}_{n,p} &= -\frac{\mathrm{e}^{2\, m\, b_0} \left[3\, m^2\, (3 w_p - 1) + 2\, b_0^2\, k^2\, \omega^2\, (3 w_p + 2) \right]}{48\, m\, b_0^2\, \omega^2\, (\mathrm{e}^{m\, b_0} - 1)} \\ c^{(1)}_{n,n} &= \frac{\mathrm{e}^{m\, b_0} \left[3\, m^2\, (3 w_n - 1) + 2\, b_0^2\, k^2\, \omega^2 \left(1 - 2 \mathrm{e}^{-m\, b_0} \right) (3 w_n + 2) \right]}{48\, m\, b_0^2\, \omega^2\, \left(\mathrm{e}^{m\, b_0} - 1 \right)} \;. \end{aligned} \qquad (37)$$

and defining the time coordinate τ as

$$\mathrm{d}\tau = \exp\left(-\frac{m\, b_0}{2} \right) \mathrm{d}t \;. \qquad (38)$$

With this choice, the cosmological Friedmann equations can be easily compared to standard ones.

Let us now comment on the first order results. As far as the radion perturbation is concerned, we found

$$\begin{aligned} \delta f_b = &\frac{m\, \mathrm{e}^{4\, m\, b_0 y}}{4\, b_0\, \omega^2} \left[(1 - 3 w_p)\, \rho_p + \mathrm{e}^{-m\, b_0} (1 - 3 w_n)\, \rho_n \right] \epsilon \\ &+ \mathcal{O}(\epsilon^2) \;, \end{aligned} \qquad (39)$$

which was expected, as it is due to the known coupling of the radion with the trace of the energy-momentum tensor of brane matter. Traceless fluids, such as radiation, have no first order effect on the excitation of the radion. If one fills the branes with some pressureless fluid, the distance between the two branes grows. This effect, being counter-intuitive for the attractive nature of Newtonian gravity, is in fact a consequence of the form of the stabilizing potential. Its non trivial contribution to the bulk energy-momentum tensor at order ϵ is

$$T_{44} \sim -2\, b_0^3\, \omega^2\, \delta f_b \;. \qquad (40)$$

The first order expressions are identical to those for a static solution, as they can be obtained by neglecting $\dot\rho_i$. Every kind of matter on the branes thus acts so as to detune the brane tensions from the bulk cosmological constant and can be balanced by some constant pressure along the fifth dimension. Such pressure is given by the first order contribution of (40) which increases when δf_b decreases.

Note that (39) is proportional to the inverse of ω^2, which represents the effective spring constant coming from some stabilization mechanism. When such a constant diverges, the correction δf_b vanishes and the length of the fifth dimension is fixed as expected, even if there is a finite, ω-independent T_{44} pressure term. On the other hand, the correction to the scale factor,

$$\delta f_a \;=\; \frac{\mathrm{e}^{2\,m\,b_0\,y}}{48\,m\,b_0^2\,\omega^2\,\left(\mathrm{e}^{m\,b_0}-1\right)}\left\{\left[2\,b_0^2\,k^2\,\omega^2\left(\mathrm{e}^{2\,m\,b_0\,y}-2\,\mathrm{e}^{m\,b_0}\right)+3\,m^2\,(3w_p-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right]\rho_p\right.$$
$$\left.+\mathrm{e}^{-m\,b_0}\,\left[2\,b_0^2\,k^2\,\omega^2\left(\mathrm{e}^{2\,m\,b_0\,y}-2\right)+3\,m^2(3w_n-1)\mathrm{e}^{2\,m\,b_0\,y}\right]\rho_n\right\}\epsilon+\mathcal{O}(\epsilon^2)\;, \tag{41}$$

never vanishes when matter is present on the branes, even if that is trace-less. Furthermore a finite, non vanishing δf_a can be obtained in the limit of infinite spring constant, regardless of the matter equation of state.

The carrection to the lapse function,

$$\delta f_n = \frac{\mathrm{e}^{2\,m\,b_0\,y}}{48\,m\,b_0^2\,\omega^2\,(\mathrm{e}^{m\,b_0}-1)}\left\{\left[2\,b_0^2\,k^2\,\omega^2\,(3w_p+2)\left(2\mathrm{e}^{m\,b_0}-\mathrm{e}^{2\,m\,b_0\,y}\right)+3\,m^2\,(3w_p-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right]\rho_p\right. \tag{42}$$
$$\left.+\mathrm{e}^{-m\,b_0}\,\left[2\,b_0^2\,k^2\,\omega^2\,(3w_n+2)\,\left(2-\mathrm{e}^{2\,m\,b_0\,y}\right)+3\,m^2(3w_n-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right]\rho_n\right\}\epsilon+\epsilon\,c^{(1)}_{n,p}\,\rho_p+\epsilon\,c^{(1)}_{n,n}\,\rho_n+\mathcal{O}(\epsilon^2)\;,$$

does not vanish when the branes are filled with trace-less matter. Note, however, that a vanishing correction can be obtained for some exotic fluid with $w_i=-2/3$ and negative pressure in the limit $\omega\to\infty$. Apart from these exceptions, one has non negligible corrections everywhere in the bulk.

In order to compare the first order results with the RS II case of a single brane, we must instead use the proper time on the positive tension brane. This is achieved by setting

$$c^{(1)}_{n,p}=\frac{\left[3\,m^2(3w_p-1)+2\,b_0^2\,k^2\,\omega^2\left(1-2\mathrm{e}^{m\,b_0}\right)(3w_p+2)\right]}{48\,m\,b_0^2\,\omega^2\,(\mathrm{e}^{m\,b_0}-1)} \tag{43}$$
$$c^{(1)}_{n,n}=\frac{\mathrm{e}^{-m\,b_0}\,\left[3\,m^2(3w_n-1)+2\,b_0^2\,k^2\,\omega^2\,(3w_n+2)\right]}{48\,m\,b_0^2\,\omega^2\,(1-\mathrm{e}^{m\,b_0})}\;,$$

and letting $b_0\to\infty$. The Friedmann equation is simply obtained by keeping $\mathcal{O}(\epsilon)$ terms in $H^2(y_i,t)$. Since the first order four-dimensional Hubble parameter is homogeneous, it reacts to all the sources along the y direction. On the positive tension brane one has, to first order in ϵ,

$$H_p^2=\frac{m\,k^2\left(\mathrm{e}^{m\,b_0}\rho_p+\mathrm{e}^{-m\,b_0}\rho_n\right)}{3\,(\mathrm{e}^{m\,b_0}-1)}\,\epsilon\;\overset{b_0\to\infty}{\longrightarrow}\;\frac{m\,k^2}{3}\,\rho_p\,\epsilon\;, \tag{44}$$

regardless of the value of ω. When the negative tension brane is moved to infinity, its contribution goes to zero and one recovers the usual first order effect in brane cosmology.

On the visible brane, the Friedmann equation is slightly modified by the rescaled time parameter τ,

$$H_n^2=\frac{m\,k^2\left(\mathrm{e}^{2\,m\,b_0}\rho_p+\rho_n\right)}{3\,(\mathrm{e}^{m\,b_0}-1)}\,\epsilon\;\overset{\rho_p\to 0}{\longrightarrow}\;\frac{m\,k^2\,\rho_n\,\epsilon}{3\,(\mathrm{e}^{m\,b_0}-1)}\;. \tag{45}$$

The acceleration equation is homogeneous as the Friedmann equation and becomes

$$\frac{\ddot a(y_i,t)}{a(y_i,t)}=\frac{m\,k^2\left[\mathrm{e}^{m\,b_0}(1+w_p)\rho_p+\mathrm{e}^{-m\,b_0}(1+w_n)\rho_n\right]}{6\,(1-\mathrm{e}^{m\,b_0})}\epsilon\;, \tag{46}$$

which has the weighted brane fluid energy densities as sources.

B. Second order results

We are now ready to evaluate $\mathcal{O}(\epsilon^2)$ corrections to the vacuum solution RS I. The procedure will be analogous to the one used for first order results in the previous section.

As a first step, one can impose the constraint (12) in order to find the dependence on y of the second order coefficients in δf_b. We are then left with three inhomogeneous equations obtained by setting to zero the coefficients of the independent matter densities in

$$f^{(2)\,\prime}_{b,p}\,\rho_p^2+f^{(2)\,\prime}_{b,n}\,\rho_n^2+f^{(2)\,\prime}_{b,m}\,\rho_p\,\rho_n-4\,m\,b_0\,\left(f^{(2)}_{b,p}\,\rho_p^2+f^{(2)}_{b,n}\,\rho_n^2+f^{(2)}_{b,m}\,\rho_p\,\rho_n\right)$$
$$+\frac{m\,(1-3\,w_p)^2\,\mathrm{e}^{6\,m\,b_0\,y}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\,\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2+b_0^2\,k^2\,\omega^2\right)-b_0^2\,k^2\,\omega^2\,\mathrm{e}^{m\,b_0}\right]\,\rho_p^2$$
$$+\frac{m\,(1-3\,w_n)^2\,\mathrm{e}^{2\,m\,b_0\,(3y-1)}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\,\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2+b_0^2\,k^2\,\omega^2\right)-b_0^2\,k^2\,\omega^2\right]\,\rho_n^2$$
$$+\frac{m\,(1-3\,w_p)(1-3\,w_n)\,\mathrm{e}^{m\,b_0\,(6y-1)}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\,\left[2\,\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2+b_0^2\,k^2\,\omega^2\right)-b_0^2\,k^2\,\omega^2\left(1+\mathrm{e}^{m\,b_0}\right)\right]\,\rho_p\,\rho_n=0\;, \tag{47}$$

which contains the first order parameters previously determined. The solutions are

$$f_{b,p}^{(2)} = \mathrm{e}^{4\,m\,b_0\,y}\left\{b_p^{(2)} - \frac{(1-3w_p)^2\,\mathrm{e}^{4\,m\,b_0\,y}}{96\,b_0^3\,\omega^4\,(\mathrm{e}^{m\,b_0}-1)^2}\left[6\,m^2 + b_0^2\,k^2\,\omega^2 - 2\,b_0^2\,k^2\,\omega^2\,\mathrm{e}^{m\,b_0\,(2y-3)}\right]\right\} \tag{48a}$$

$$f_{b,n}^{(2)} = \mathrm{e}^{4\,m\,b_0\,y}\left\{b_n^{(2)} - \frac{(1-3w_n)^2\,\mathrm{e}^{6\,m\,b_0\,(y-1)}}{96\,b_0^3\,\omega^4\,(\mathrm{e}^{m\,b_0}-1)^2}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - 2\,b_0^2\,k^2\,\omega^2\right]\right\} \tag{48b}$$

$$f_{b,m}^{(2)} = \mathrm{e}^{4\,m\,b_0\,y}\left\{b_m^{(2)} - \frac{(1-3w_p)\,(1-3w_n)\,\mathrm{e}^{m\,b_0\,(6y-5)}}{48\,b_0^3\,\omega^4\,(\mathrm{e}^{m\,b_0}-1)^2}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - b_0^2\,k^2\,\omega^2\left(1+\mathrm{e}^{m\,b_0}\right)\right]\right\}\,, \tag{48c}$$

where

$$b_p^{(2)}\,,\quad b_n^{(2)}\,,\quad b_m^{(2)}\,, \tag{49}$$

are integration constants to be determined from the junction conditions. Once we plug the $f_{b,i}^{(2)}$'s into the second order terms in Eq. (11), we get the equations for the $f_{a,i}^{(2)}$'s, which are not displayed for the sake of brevity. Finally, by solving $G_{44} = k^2\,T_{44}$ one obtains the corrections $f_{n,i}^{(2)}$'s. The results will contain six integration constants from the solutions of the first order differential equations for $f_{a,i}^{(2)}$ and $f_{n,i}^{(2)}$,

$$c_{a,i}^{(2)}\,,\quad c_{n,i}^{(2)}\,, \tag{50}$$

and nine parameters related to the radion, the four-dimensional Hubble parameter and $c(t)$ respectively,

$$h_{h,i}^{(2)}\,,\quad b_i^{(2)}\,,\quad c_i^{(2)}\,, \tag{51}$$

where i runs over p, n and m for second order quantities. Analogously to the first order case, one can fix the coefficients (51) by imposing the junction conditions, which form a system of nine independent equations. Nonetheless, one is again not able to fix the constants (50). Here are the solutions obtained at the end of the calculations described above:

$$\begin{aligned}
f_{a,p}^{(2)} \;=\; & \frac{1}{4608\,m^2\,b_0^4\,\omega^4\,(\mathrm{e}^{m\,b_0}-1)^2}\Big\{2\,\mathrm{e}^{4\,m\,b_0\,y}\left[3\,m^2(1-3w_p)^2 - 8\,b_0^2\,k^2\,\omega^2\right]\left(6\,m^2 - b_0^2\,k^2\omega^2\right)\\
&+8\,b_0^2\,k^2\,\omega^2\,\mathrm{e}^{m\,b_0\,(6y+1)}\left[11\,m^2\,(3w_p-1) + 2\,b_0^2\,k^2\omega^2\right]\\
&-4\,b_0^2\,k^2\,\omega^2\,\mathrm{e}^{m\,b_0\,(2y+1)}\left[3\,m^2\,(9\,w_p^2 + 18\,w_p - 7) + 8\,b_0^2\,k^2\omega^2\right]\\
&+4\,b_0^2\,k^2\omega^2\,\mathrm{e}^{2\,m\,b_0\,(y+1)}\left[27\,m^2\,(3w_p^2 + 2w_p - 1) + 16\,b_0^2\,k^2\omega^2\right] +\\
&+\mathrm{e}^{8\,m\,b_0\,y}\left[45\,m^4(1-3w_p)^2 + 3\,m^2 b_0^2 k^2\omega^2(9w_p^2 - 42w_p + 13) - 4b_0^4k^4\omega^4\right]\\
&-2\,\mathrm{e}^{2\,m\,b_0\,(2y+1)}\left[81m^4(9w_p^3 + 9w_p^2 - w_p - 1) + 3m^2b_0^2k^2\omega^2(27w_p^2 + 42w_p - 1) + 8b_0^4k^4\omega^4\right]\\
&-2\mathrm{e}^{m\,b_0\,(4y+1)}\left[81\,m^4(9w_p^3 + 9w_p^2 - w_p - 1) - 3\,m^2b_0^2k^2\omega^2(9w_p^2 + 6w_p + 29) + 16\,b_0^4k^4\omega^4\right]\Big\} + c_{a,p}^{(2)}
\end{aligned} \tag{52a}$$

$$\begin{aligned}
f_{a,n}^{(2)} \;=\; & \frac{\mathrm{e}^{2mb_0(y-1)}}{4608m^2b_0^4\omega^4\,(\mathrm{e}^{mb_0}-1)^2}\Big\{2\mathrm{e}^{2mb_0(y+1)}\left[3m^2(1-3w_n)^2 - 8b_0^2k^2\omega^2\right]\left(6m^2 - b_0^2k^2\omega^2\right)\\
&+8b_0^2k^2\omega^2\mathrm{e}^{4mb_0y}\left[11m^2(3w_n-1) + 2b_0^2k^2\omega^2\right] - 4b_0^2k^2\omega^2\mathrm{e}^{2mb_0}\left[3m^2(9w_n^2 + 18w_n - 7) + 8b_0^2k^2\omega^2\right]\\
&+4b_0^2k^2\omega^2\mathrm{e}^{mb_0}\left[27m^2(3w_n^2 + 2w_n - 1) + 16b_0^2k^2\omega^2\right]\\
&+\mathrm{e}^{6mb_0y}\left[45m^4(1-3w_n)^2 + 3m^2b_0^2k^2\omega^2(9w_n^2 - 42w_n + 13) - 4b_0^4k^4\omega^4\right]\\
&-2\mathrm{e}^{2mb_0y}\left[81m^4(9w_n^3 + 9w_n^2 - w_n - 1) + 3m^2b_0^2k^2\omega^2(27w_n^2 + 42w_n - 1) + 8b_0^4k^4\omega^4\right]\\
&-2\mathrm{e}^{mb_0(2y+1)}\left[81m^4(9w_n^3 + 9w_n^2 - w_n - 1) - 3m^2b_0^2k^2\omega^2(9w_n^2 + 6w_n + 29) + 16b_0^4k^4\omega^4\right]\Big\} + c_{a,n}^{(2)}
\end{aligned} \tag{52b}$$

$$
\begin{aligned}
f^{(2)}_{a,m} &= \frac{\mathrm{e}^{mb_0(2y-1)}}{2304m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{4b_0^2k^2\omega^2\mathrm{e}^{4mb_0y}\left[m^2(3w_p-1)(6w_p-6w_n+11)+2b_0^2k^2\omega^2\right] \\
&+4b_0^2k^2\omega^2\mathrm{e}^{mb_0(4y+1)}\left[m^2(3w_n-1)(6w_n-6w_p+11)+2b_0^2k^2\omega^2\right] \\
&+4b_0^2k^2\omega^2\mathrm{e}^{mb_0}\left[3m^2(18w_p^2-9w_pw_n+9w_p-9w_n-1)+4b_0^2k^2\omega^2\right] \\
&+4b_0^2k^2\omega^2\mathrm{e}^{2mb_0}\left[3m^2(18w_n^2-9w_pw_n+9w_n-9w_p-1)+4b_0^2k^2\omega^2\right] \\
&+\mathrm{e}^{6mb_0y}\left[45m^4(3w_p-1)(3w_n-1)+3m^2b_0^2k^2\omega^2(9w_pw_n-21w_p-21w_n+13)-4b_0^4k^4\omega^4\right] \\
&-\mathrm{e}^{2mb_0y}\big[9m^4(3w_p-1)(18w_p^2+9w_pw_n+27w_p+3w_n+11) \\
&+36m^2b_0^2k^2\omega^2(3w_p^2-3w_pw_n+w_p-2w_n-1)+8b_0^4k^4\omega^4\big] \\
&-\mathrm{e}^{2mb_0(y+1)}\big[9m^4(3w_n-1)(18w_n^2+9w_pw_n+27w_n+3w_n+11) \\
&+36m^2b_0^2k^2\omega^2(3w_n^2-3w_pw_n+w_n-2w_p-1)+8b_0^4k^4\omega^4\big] \\
&-\mathrm{e}^{mb_0(2y+1)}\big[81m^4(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+7w_p^2+7w_n^2+4w_pw_n-w_p-w_n-2) \\
&+6m^2b_0^2k^2\omega^2(18w_p^2+18w_n^2-9w_pw_n+21w_p+21w_n-1)+16b_0^4k^4\omega^4\big]\Big\}+c^{(2)}_{a,m}
\end{aligned}
\tag{52c}
$$

$$
\begin{aligned}
f^{(2)}_{n,p} &= \frac{\mathrm{e}^{2mb_0y}}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{-8b_0^2k^2\omega^2(3w_p+2)\mathrm{e}^{3mb_0}\left[3m^2(3w_p-1)+2b_0^2k^2\omega^2(3w_p+2)\right] \\
&-8b_0^2k^2\omega^2\mathrm{e}^{mb_0(4y+1)}\left[m^2(81w_p^3+180w_p^2+24w_p-31)+2b_0^2k^2\omega^2(6w_p+5)\right] \\
&+2\mathrm{e}^{2mb_0y}\left(6m^2-b_0^2k^2\omega^2\right)\left[3m^2(3w_p-1)^2-4b_0^2k^2\omega^2(9w_p^2+6w_p-1)\right] \\
&-4b_0^2k^2\omega^2\mathrm{e}^{mb_0}\left[-3m^2(63w_p^2+30w_p-17)+4b_0^2k^2\omega^2(9w_p^2+6w_p-1)\right] \\
&+4b_0^2k^2\omega^2\left[-27m^2(18w_p^3+27w_p^2+4w_p-5)+8b_0^2k^2\omega^2(9w_p^2+6w_p-1)\right] \\
&+\mathrm{e}^{6mb_0y}\left[45m^4(3w_p-1)^2+3m^2b_0^2k^2\omega^2(117w_p^2+30w_p-23)+4b_0^4k^4\omega^4(6w_p+5)\right] \\
&-2\mathrm{e}^{mb_0(2y+1)}\big[81m^4(9w_p^3+9w_p^2-w_p-1)-3m^2b_0^2k^2\omega^2(162w_p^3+279w_p^2+60w_p-25) \\
&+8b_0^4k^4\omega^4(9w_p^2+6w_p-1)\big]+2\mathrm{e}^{2mb_0(y+1)}\big[-9m^4(81w_p^3+90w_p^2-15w_p-8) \\
&+3m^2b_0^2k^2\omega^2(162w_p^3+243w_p^2+12w_p-53)+4b_0^4k^4\omega^4(9w_p^2+24w_p+14)\big]\Big\}+c^{(2)}_{n,p}
\end{aligned}
\tag{52d}
$$

$$
\begin{aligned}
f^{(2)}_{n,n} &= \frac{\mathrm{e}^{2mb_0(y-1)}}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{-4b_0^2k^2\omega^2(6w_n+5)\mathrm{e}^{mb_0}\left[27m^2(3w_n^2+2w_n-1)+8b_0^2k^2\omega^2\right] \\
&-8b_0^2k^2\omega^2\mathrm{e}^{4mb_0y}\left[m^2(81w_n^3+180w_n^2+24w_n-31)+2b_0^2k^2\omega^2(6w_n+5)\right] \\
&+4b_0^2k^2\omega^2\mathrm{e}^{2mb_0}\left[3m^2(45w_n^2+24w_n-13)+4b_0^2k^2\omega^2(6w_n+5)\right] \\
&-2\mathrm{e}^{mb_0(2y+1)}\big[81m^4(9w_n^3+9w_n^2-w_n-1)-3m^2b_0^2k^2\omega^2(162w_n^3+243w_n^2+48w_n-17) \\
&-8b_0^4k^4\omega^4(6w_n+5)\big]+2\mathrm{e}^{2mb_0(y+1)}\big[9m^4(3w_n-1)^2 \\
&-3m^2b_0^2k^2\omega^2(45w_n^2+30w_n+1)-4b_0^4k^4\omega^4(6w_n+5)\big]+\mathrm{e}^{6mb_0y}\big[45m^4(3w_n-1)^2 \\
&+3m^2b_0^2k^2\omega^2(117w_n^2+30w_n-23)+4b_0^4k^4\omega^4(6w_n+5)\big]+2\mathrm{e}^{2mb_0y}\big[-81m^4(9w_n^3+9w_n^2-w_n-1) \\
&+3m^2b_0^2k^2\omega^2(162w_n^3+243w_n^2+12w_n-53)+8b_0^4k^4\omega^4(6w_n+5)\big]\Big\}+c^{(2)}_{n,n}
\end{aligned}
\tag{52e}
$$

$$
\begin{aligned}
f_{n,m}^{(2)} = & \frac{\mathrm{e}^{-mb_0}}{2304m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{4b_0^2k^2\omega^2(3w_p+2)\mathrm{e}^{mb_0(2y+3)}\left[3m^2(1-3w_n)+2b_0^2k^2\omega^2(3w_n+2)\right] \\
& -4b_0^2k^2\omega^2\mathrm{e}^{6mb_0y}\left[m^2(3w_p-1)(18w_p^2+9w_pw_n+45w_p+24w_n+31)+2b_0^2k^2\omega^2(3w_p+3w_n+5)\right] \\
& -4b_0^2k^2\omega^2\mathrm{e}^{mb_0(6y+1)}\left[m^2(3w_n-1)(18w_n^2+9w_pw_n+45w_n+24w_p+31)+2b_0^2k^2\omega^2(3w_p+3w_n+5)\right] \\
& -4b_0^2k^2\omega^2\mathrm{e}^{2mb_0(y+1)}\left[3m^2(3w_n+2)(18w_n^2+9w_pw_n-12w_p+27w_n-8)\right. \\
& \left.+2b_0^2k^2\omega^2(18w_pw_n+15w_p+15w_n+13)\right]-4b_0^2k^2\omega^2\mathrm{e}^{mb_0(2y+1)}\left[-2b_0^2k^2\omega^2(9w_pw_n+3w_p+3w_n-1)\right. \\
& \left.+3m^2(3w_p+2)(18w_p^2+9w_pw_n+27w_p-15w_n-7)\right]+\mathrm{e}^{8mb_0y}\left[45m^4(3w_p-1)(3w_n-1)\right. \\
& \left.+3m^2b_0^2k^2\omega^2(117w_pw_n+15w_p+15w_n-23)+4b_0^4k^4\omega^4(3w_p+3w_n+5)\right] \\
& +\mathrm{e}^{mb_0(4y+1)}\left[-81m^4(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+7w_p^2+7w_n^2+4w_pw_n-w_p-w_n-2)\right. \\
& +6m^2b_0^2k^2\omega^2(54w_p^3+54w_n^3+27w_p^2w_n+27w_pw_n^2+117w_p^2+117w_n^2-9w_pw_n-6w_p+12w_n-49) \\
& \left.+8b_0^4k^4\omega^4(9w_pw_n+12w_p+12w_n+14)\right]+\mathrm{e}^{4mb_0y}\left[-9m^4(3w_p-1)(18w_p^2+9w_pw_n+27w_p+3w_n+11)\right. \\
& \left.+54m^2b_0^2k^2\omega^2(6w_p^3+3w_p^2w_n+13w_p^2-2w_pw_n+4w_p-3w_n-1)-4b_0^4k^4\omega^4(9w_pw_n+3w_p+3w_n-1)\right] \\
& +\mathrm{e}^{2mb_0(2y+1)}\left[-81m^4(w_p+2w_n+1)(3w_n^2+2w_n-1)\right. \\
& +6m^2b_0^2k^2\omega^2(54w_n^3+27w_pw_n^2+117w_n^2-15w_p+30w_n-13) \\
& \left.-4b_0^4k^4\omega^4(9w_pw_n+3w_p+3w_n-1)\right]\Big\}+c_{n,m}^{(2)}\ .
\end{aligned} \tag{52f}
$$

IV. SECOND ORDER COSMOLOGY

Let us now look at the physical consequences of the above results as applied to two different cosmological scenarios. The Friedmann and acceleration equations will be showed and their phenomenology investigated in different regimes of ω and for different matter equations of state on the branes.

A. RS I

We begin from the case with two branes [14] for which we shall study how cosmology would be described by observers on the negative brane. In order to achieve that, one can use the corresponding proper time by fixing the $c_{n,i}^{(1)}$'s as in Eq. (37), choosing

$$
\begin{aligned}
c_{n,p}^{(2)} = & \frac{\mathrm{e}^{2mb_0}}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{9m^4(3w_p-1)\left[3\mathrm{e}^{2mb_0}(3w_p^2+4w_p+1)+18\mathrm{e}^{mb_0}(3w_p^2+4w_p+1)-12w_p+4\right] \\
& -m^2b_0^2k^2\omega^2\left[\mathrm{e}^{2mb_0}(324w_p^3+153w_p^2-102w_p-91)-6\mathrm{e}^{mb_0}(162w_p^3+207w_p^2+12w_p-65)\right. \\
& \left.+54(5w_p^2+2w_p-3)\right]+4b_0^4k^4\omega^4\left[3\mathrm{e}^{2mb_0}(6w_p^2+6w_p+1)+2\left(1-2\mathrm{e}^{mb_0}\right)(9w_p^2+6w_p-1)\right]\Big\}
\end{aligned} \tag{53a}
$$

$$
\begin{aligned}
c_{n,n}^{(2)} = & \frac{1}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{9m^4(3w_n-1)\left[-7\mathrm{e}^{2mb_0}(3w_n-1)+18\left(\mathrm{e}^{mb_0}+1\right)(3w_n^2+4w_n+1)\right] \\
& -m^2b_0^2k^2\omega^2\left[\mathrm{e}^{2mb_0}(81w_n^2-90w_n-75)+2\mathrm{e}^{mb_0}(162w_n^3+279w_n^2+192w_n-5)\right. \\
& \left.-6(162w_n^3+243w_n^2+60w_n-37)\right]+4b_0^4k^4\omega^4\left(\mathrm{e}^{mb_0}-2\right)^2(6w_n+5)\Big\}
\end{aligned} \tag{53b}
$$

$$
\begin{aligned}
c_{n,m}^{(2)} = & \frac{\mathrm{e}^{mb_0}}{2304m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\Big\{9m^4\left[\mathrm{e}^{2mb_0}(3w_n-1)(18w_n^2+9w_pw_n-6w_p+27w_n+14)\right. \\
& 9\mathrm{e}^{mb_0}(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+7w_p^2+7w_n^2+4w_pw_n-w_p-w_n-2)+(3w_p-1)(18w_p^2 \\
& \left.+9w_pw_n+27w_p+3w_n+11)\right]-m^2b_0^2k^2\omega^2\left[\mathrm{e}^{2mb_0}(108w_n^3+54w_pw_n^2+234w_n^2-9w_pw_n+87w_p\right. \\
& -39w_n+1)+2\mathrm{e}^{mb_0}(54w_p^3-162w_n^3+27w_p^2w_n-81w_pw_n^2+117w_p^2-351w_n^2-45w_pw_n+30w_p \\
& \left.-96w_n+11)-6(54w_p^3+27w_p^2w_n+117w_p^2-36w_pw_n+30w_p-33w_n-19)\right] \\
& -4b_0^4k^4\omega^4\left[\left(\mathrm{e}^{2mb_0}-2\mathrm{e}^{mb_0}\right)(3w_p+2)(3w_n+2)+9w_pw_n+3w_p+3w_n-1\right]\Big\}
\end{aligned} \tag{53c}
$$

and finally rescaling $n(y,t)$ to satisfy the condition $n(1/2,\tau)=1$. The Friedmann equation we are interested in is given by the second order expression of the Hubble parameter at $y=1/2$ as a function of τ,

$$\begin{aligned} H^2(1/2,\tau) \;=\; & \frac{k^2 m}{3\left(\mathrm{e}^{mb_0}-1\right)}\left(\rho_n+\mathrm{e}^{2mb_0}\rho_p\right)-(w_n+1)\,(3w_n-1)\,\frac{3m^2k^2\left(\mathrm{e}^{mb_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\,\rho_n^2 \\ & -\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{mb_0}+1\right)}{48b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[18w_p^2-9w_p(w_n-1)-9w_n-1+2\mathrm{e}^{mb_0}(9w_n^2+9w_n-4)\right]\right\}\mathrm{e}^{mb_0}\,\rho_p\,\rho_n \\ & +\left\{\frac{k^4}{36}+\frac{m^2k^2(3w_p-1)\left(\mathrm{e}^{mb_0}+1\right)}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[3w_p+7-4\mathrm{e}^{mb_0}(3w_p+4)\right]\right\}\mathrm{e}^{2mb_0}\,\rho_p^2\,. \end{aligned} \tag{54}$$

Note that the results (53a)-(54) do not depend on the integration constants $c_{a,i}^{(1)}$'s and $c_{a,i}^{(2)}$'s, which reflects the fact that the three-dimensional spatial curvature has been set to zero ab initio.

The Friedmann equation contains coefficients up to second order in the vacuum perturbations of both branes. First and second order contributions in Eq. (54) are consequences of the adiabatic regime of the five-dimensional dynamics which determines the value of the integration constant $\tilde{c}(t)$ in Eq. (11). The value of the latter is affected by the presence of matter on both branes through the junction conditions and, for instance, up to $\mathcal{O}(\epsilon^2)$ is given by

$$\lim_{\omega\to\infty} c^{(1)} \;=\; \frac{m\,k^2\,\left(\rho_p+\mathrm{e}^{mb_0}\rho_n\right)}{3\,\left(\mathrm{e}^{m\,b_0}-1\right)} \tag{55a}$$

$$\lim_{\omega\to\infty} c^{(2)} \;=\; \frac{k^4}{36}\left[\frac{\mathrm{e}^{2\,m\,b_0}+2\,\mathrm{e}^{m\,b_0}-1}{\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_p^2+\left(\frac{\mathrm{e}^{m\,b_0}+1}{\mathrm{e}^{m\,b_0}-1}\right)^2\mathrm{e}^{-m\,b_0}\,\rho_p\rho_n+\frac{\mathrm{e}^{-2\,m\,b_0}+2\,\mathrm{e}^{-m\,b_0}-1}{\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2\right]\,, \tag{55b}$$

in the limit of infinite spring constant. Furthermore, as previously noted, this effect is also a consequence of the radion field potential acting as a source in Eq. (11). One can expect to cancel some terms in Eq. (54) by arbitrarily increasing the spring constant of the effective radion potential in order to decrease the radion shift from equilibrium. This mechanism partially works as the first order contributions to the bulk potential vanish along the time-time direction, whereas only second order terms, which depend on the matter equation of state, cancel when $\omega\to\infty$. One is then left with

$$\begin{aligned} \lim_{\omega\to\infty} H^2(1/2,\tau) = & \frac{m\,k^2\,\left(\rho_n+\mathrm{e}^{2\,m\,b_0}\rho_p\right)}{3\,\left(\mathrm{e}^{mb_0}-1\right)} \\ & -\frac{k^4}{36}\,\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n+\frac{k^4}{36}\,\mathrm{e}^{2\,m\,b_0}\,\rho_p^2\,, \end{aligned} \tag{56}$$

which is analogous to what has been obtained in Ref. [10]. The matter on the positive tension brane appears at second order with the role of some "dark" fluid and acts as a sort of Brans-Dicke field which adiabatically modifies the Newton constant perceived in the visible Universe. This behavior is somehow inherited by the dynamics of the radion which is known to modulate the strength of gravity on the visible brane.

Note that while first order coefficients are positive definite, irrespective of the brane tension, the sign of second order ones depends on the matter equation of state. Differently enough from unstabilized brane cosmology, this fact implies that leading order cosmological equations have the correct behavior on both branes. Letting the energy density $\rho_p\to 0$ in Eq. (54), one obtains

$$\begin{aligned} H^2(1/2,\tau) = & (w_n+1)\,(1-3w_n)\,\frac{3\,m^2\,k^2\,\left(\mathrm{e}^{m\,b_0}+1\right)}{32\,b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2 \\ & +\frac{m\,k^2\,\rho_n}{3\,\left(\mathrm{e}^{m\,b_0}-1\right)}\,, \end{aligned} \tag{57}$$

which has the usual first order solution for both radiation and a cosmological constant on the visible brane. A matter dominated Universe would otherwise generate second order corrections.

We now come to the equation for the acceleration, which has the general form

$$\frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = \frac{m\,k^2}{6\,(1-\mathrm{e}^{m\,b_0})}\left[(3w_n+1)\rho_n + \mathrm{e}^{2\,m\,b_0}(3w_p+1)\rho_p\right] + (w_n+1)\,(3w_n-1)\,(3w_n+2)\,\frac{3m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\rho_n^2$$
$$-\left\{\frac{k^4}{144}(3w_n+1)(3w_p+1) + \frac{m^2k^2}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\left[\mathrm{e}^{2\,m\,b_0}(3w_n-1)(18w_n^2+9w_pw_n+15w_p+39w_n+23)\right.\right.$$
$$+9\mathrm{e}^{m\,b_0}(6w_p^3+w_n^3+3w_p^2w_n+3w_pw_n^2+11w_p^2+11w_n^2+2w_pw_n-4)$$
$$\left.\left.+54w_p^3-27w_p^2w_n+99w_p^2-18w_pw_n+24w_p-13\right]\right\}\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n$$
$$+\left\{\frac{k^4}{144}(3w_p+1)^2 + \frac{m^2k^2(3w_p-1)\left(\mathrm{e}^{m\,b_0}+1\right)}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\left[\mathrm{e}^{m\,b_0}(27w_p^2+54w_p+23)-9w_p-5\right]\right\}\mathrm{e}^{2\,m\,b_0}\rho_p^2\ , \qquad (58)$$

and, for $\omega \to \infty$, reduces to

$$\frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = \frac{m\,k^2}{6\,(1-\mathrm{e}^{m\,b_0})}\left[(3w_n+1)\rho_n + \mathrm{e}^{2\,m\,b_0}(3w_p+1)\,\rho_p\right]$$
$$-\frac{k^4}{144}(3w_n+1)(3w_p+1)\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n + \frac{k^4}{144}(3w_p+1)^2\mathrm{e}^{2\,m\,b_0}\,\rho_p^2\ , \qquad (59)$$

which is again analogous to the result of Ref. [10]. The coefficient of ρ_p^2 in Eq. (59) is positive or zero in this limit and provides an accelerating contribution to the equation. The coefficient of the mixed term has a positive value when just one fluid has $w_i < -1/3$. For $\rho_p \to 0$, one is further left with

$$\frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = \frac{m\,k^2\,(3w_n+1)}{6\,(1-\mathrm{e}^{mb_0})}\,\rho_n + (w_n+1)\,(3w_n-1)\,(3w_n+2)\,\frac{3m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2\ , \qquad (60)$$

where the second order contribution is inversely proportional to ω and vanishes for radiation and a cosmological constant. This peculiarity leads to the standard cosmological evolution up to $\mathcal{O}(\rho_n^2)$ until the matter dominated era.

A singularity in the lapse function $n(y,t)$ for $b_0 \to \infty$ prevents us from analyzing the correct limit when the distance between the branes becomes infinite, we shall thus comment on this problem in the next subsection. On setting $\rho_p \to 0$, Eq. (56) admits the finite but trivial limit

$$\lim_{b_0\to\infty} H^2(1/2,\tau) = 0\ , \qquad (61a)$$

in which one also has

$$\lim_{b_0\to\infty}\frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = 0\ . \qquad (61b)$$

This result is due to the reduced strength of the gravitational interaction at infinity.

B. RS II

On can think of the RS II model as the limit of RS I in which the distance between the two branes becomes infinite, thus one expects that only ρ_p contributes in this limit. The cosmological proper time is now the one on the Planck brane and is recovered upon choosing

$$c_{n,p}^{(2)} = \frac{1}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\left\{9m^4(3w_p-1)\left[18\mathrm{e}^{2mb_0}(3w_p^2+4w_p+1)+18\mathrm{e}^{mb_0}(3w_p^2+4w_p+1)\right.\right.$$
$$\left.-21w_p+7\right]+m^2b_0^2k^2\omega^2\left[6\mathrm{e}^{2mb_0}(162w_p^3+243w_p^2+60w_p-37)\right.$$
$$\left.-2\mathrm{e}^{mb_0}(162w_p^3+279w_p^2+192w_p-5)-81w_p^2+90w_p+75\right]$$
$$\left.+4b_0^4k^4\omega^4\left(1-2\mathrm{e}^{mb_0}\right)^2(6w_p+5)\right\} \qquad (62a)$$

$$c^{(2)}_{n,n} = \frac{\mathrm{e}^{-2mb_0}}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\left\{9m^4(3w_n-1)\left[4\mathrm{e}^{2mb_0}(1-3w_n)+18\mathrm{e}^{mb_0}(3w_n^2+4w_n+1)\right.\right.$$
$$\left.+54w_n^2+63w_n+21\right]-m^2b_0^2k^2\omega^2\left[54\mathrm{e}^{2mb_0}(5w_n^2+2w_n-3)\right.$$
$$\left.-6\mathrm{e}^{mb_0}(162w_n^3+207w_n^2+12w_n-65)+324w_n^3+153w_n^2-102w_n-91\right]$$
$$\left.+4b_0^4k^4\omega^4\left[2\left(\mathrm{e}^{2mb_0}-2\mathrm{e}^{mb_0}\right)(9w_n^2+6w_n-1)+3(6w_n^2+6w_n+1)\right]\right\} \tag{62b}$$

$$c^{(2)}_{n,m} = \frac{\mathrm{e}^{mb_0}}{2304m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\left\{9m^4\left[\mathrm{e}^{2mb_0}(3w_n-1)(18w_n^2+9w_pw_n+3w_p+27w_n+11)\right.\right.$$
$$+9\mathrm{e}^{mb_0}(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+7w_p^2+7w_n^2+4w_pw_n-w_p-w_n-2)$$
$$\left.(3w_p-1)(18w_p^2+9w_pw_n+27w_p-6w_n+14)\right]+m^2b_0^2k^2\omega^2\left[6\mathrm{e}^{2mb_0}(54w_n^2++27w_pw_n^2\right.$$
$$+117w_n^2-36w_pw_n-33w_p+30w_n-19)+2\mathrm{e}^{mb_0}(162w_p^3-54w_n^3+81w_p^2w_n-27w_pw_n^2$$
$$\left.+351w_p^2-117w_n^2+96w_p-30w_n-11)-108w_p^3-54w_p^2w_n-234w_p^2+9w_pw_n+39w_p-87w_n-1\right]$$
$$\left.-4b_0^4k^4\omega^4\left[\mathrm{e}^{2mb_0}(9w_pw_n+3w_p+3w_n-1)+\left(1-2\mathrm{e}^{mb_0}\right)(3w_p+2)(3w_n+2)\right]\right\} \tag{62c}$$

together with Eq. (43). The Friedmann equation, for a finite b_0, reduces to

$$H^2(0,t) = \frac{mk^2}{3\left(e^{m\,b_0}-1\right)}\left(\mathrm{e}^{m\,b_0}\rho_p+\mathrm{e}^{-m\,b_0}\rho_n\right)+(w_p+1)(1-3w_p)\frac{3m^2k^2\mathrm{e}^{m\,b_0}\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\rho_p^2$$
$$-\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)\mathrm{e}^{m\,b_0}(18w_n^2-9w_pw_n-9w_p+9w_n-1)(9w_p^2+9w_p-4)}{24b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\right\}\mathrm{e}^{-m\,b_0}\rho_p\,\rho_n$$
$$+\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)(3w_n-1)\left[\mathrm{e}^{m\,b_0}(3w_n+7)-4(3w_n+4)\right]}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\right\}\mathrm{e}^{-2\,m\,b_0}\rho_n^2\ , \tag{63}$$

which is similar to Eq. (54) but has the finite, non trivial limit

$$\lim_{b_0\to\infty}H^2(0,t)=\frac{m\,k^2}{3}\,\rho_p\ . \tag{64}$$

This is precisely the standard Friedmann equation one has in four-dimensional cosmology. Furthermore, the equation for the acceleration to second order in terms of the time on the positive tension brane is given by

$$\frac{\ddot a(0,t)}{a(0,t)}=\frac{m\,k^2\,\mathrm{e}^{m\,b_0}}{6\left(1-\mathrm{e}^{m\,b_0}\right)}\left[(3w_p+1)\rho_p+(3w_n-1)\mathrm{e}^{-2\,m\,b_0}\rho_n\right]+(w_p+1)(3w_p-1)(3w_p+2)\frac{3m^2k^2\mathrm{e}^{m\,b_0}\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\rho_p^2$$
$$-\left\{\frac{k^4}{144}(3w_p+1)(3w_n+1)-\frac{m^2k^2}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[\mathrm{e}^{2mb_0}(54w_n^3+27w_pw_n^2+99w_n^2-18w_pw_n\right.\right.$$
$$-21w_p+24w_n-13)+9\mathrm{e}^{mb_0}(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+11w_p^2+11w_n^2+2w_pw_n$$
$$\left.\left.+w_p+w_n-4)+(3w_p-1)(18w_p^2+9w_pw_n+39w_p+15w_n+23)\right]\right\}\mathrm{e}^{-mb_0}\rho_p\rho_n$$
$$+\left\{\frac{k^4}{144}(3w_n+1)^2-\frac{m^2k^2\left(\mathrm{e}^{mb_0}\right)(3w_n-1)}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[\mathrm{e}^{mb_0}(9w_n+5)-27w_n^2+54w_n+23\right]\right\}\mathrm{e}^{-2mb_0}\rho_n^2\ , \tag{65}$$

and, compatibly with Eq. (64), yields

$$\lim_{b_0\to\infty}\frac{\ddot a(0,t)}{a(0,t)}=-\frac{m\,k^2}{6}(3w_p+1)\,\rho_p\ , \tag{66}$$

in the limit of infinite brane distance. Thus second order effects, typical of brane cosmology, become more and more negligible when the distance between the branes

grows, which consequently leads to unobservable deviations from standard four-dimensional General Relativity.

V. APPROXIMATION ANALYSIS

The results obtained so far hold with the assumption that $\rho_i \ll M^4$ where M is, in general, the natural mass scale of the model. By taking all the mass parameters to such a natural scale, one expects the solutions to second order provide a good approximation for Eq. (13). In this regime, however, second order effects are certainly subleading and thus insufficient to significantly alter first order behavior. We shall hence present below a numerical analysis of the validity of our approximate solutions in the attempt to extend the range of the parameters in which ours results hold valid and widen the conclusions one can draw from second order expressions. In particular, we are interested in possible deviations from standard cosmological equations due to terms of $\mathcal{O}(\epsilon^2)$ in the first and third of Eqs. (8). Note that all numerical results will be obtained by setting the expansion parameter $\epsilon = 1$ as previously prescribed.

In order to test our approximations, since Eqs. (13) are not analytically solvable, we substitute the second order solutions into the exact Einstein equations to obtain an estimate of their non vanishing remainders which we then compare with the leading contributions (satisfying the corresponding approximate equations). Since the time dependence is contained in ρ_i in our expansions (20a)-(20c), it is also convenient to trade the time for ρ. We thus divide Eq. (11) into the following five terms

$$\eta_1(y,\rho) \equiv \frac{1}{n^2(y,t)}\frac{\dot{a}^2(y,t)}{a^2(y,t)} \tag{67a}$$

$$\eta_2(y,\rho) \equiv -\frac{1}{b^2(y,t)}\left(\frac{a'(y,t)}{a(y,t)}\right)^2 \tag{67b}$$

$$\eta_3(y,\rho) \equiv -\frac{k^2}{6}(\Lambda+U) \tag{67c}$$

$$\eta_4(y,\rho) \equiv \frac{k^2}{6\,a^4(y,t)}\int^y a^4\,\left(T^0{}_0\right)'\,\mathrm{d}x \tag{67d}$$

$$\eta_5(y,\rho) \equiv -\frac{\tilde{c}(t)}{a^4(y,t)}\ , \tag{67e}$$

in which $t = t(\rho)$ in the right hand sides is understood as the time when $\rho = \rho_n$ (ρ_p will be chosen either equal to ρ_n or zero). The sum,

$$R_\eta(y,\rho) \equiv \sum_l \eta_l\ , \tag{68}$$

evaluated on the second order solutions yields $R_\eta(y,\rho) = \mathcal{O}(\epsilon^3)$ as a measure of the corresponding error. Similarly, the third of Eqs. (13) may be written as the sum of the following six terms

$$\xi_1(y,\rho) \equiv \frac{1}{n^2(y,t)}\frac{\ddot{a}(y,t)}{a(y,t)} \tag{69a}$$

$$\xi_2(y,\rho) \equiv -\frac{1}{b^2(y,t)}\left(\frac{a'(y,t)}{a(y,t)}\right)^2 \tag{69b}$$

$$\xi_3(y,\rho) \equiv -\frac{1}{b^2(y,t)}\frac{a'(y,t)}{a(y,t)}\frac{n'(y,t)}{n(y,t)} \tag{69c}$$

$$\xi_4(y,\rho) \equiv \frac{1}{n^2(y,t)}\left(\frac{\dot{a}^2(y,t)}{a^2(y,t)} - \frac{\dot{a}(y,t)\,\dot{n}(y,t)}{a(y,t)\,n(y,t)}\right) \tag{69d}$$

$$\xi_5(y,\rho) \equiv -\frac{k^2}{3}(\Lambda+U) \tag{69e}$$

$$\xi_6(y,\rho) \equiv -\frac{2}{3}k^2\,\omega^2\,b(y,t)\,\left[b(y,t)-b_0\right]\ , \tag{69f}$$

and

$$R_\xi(y,\rho) \equiv \sum_l \xi_l\ , \tag{70}$$

with $R_\xi(y,\rho) = \mathcal{O}(\epsilon^3)$ for the same approximate solutions. One may now assume that approximate metric functions computed to $\mathcal{O}(\epsilon^2)$ are accurate approximations of exact solutions to Eqs. (13) if

$$|\eta_l| \gg |R_1| \quad \text{and} \quad |\xi_l| \gg |R_2|\ , \tag{71}$$

for every term in Eqs. (67a)-(69f) evaluated to $\mathcal{O}(\epsilon^2)$. (It should actually be sufficient to satisfy the above conditions for the leading terms of each equation.) Throughout this section, where unspecified, a natural choice [15] of dimensionful parameters is considered.

Fig. 1 shows the functions η_l and ξ_l evaluated to second order in ϵ and the corresponding R_η and R_ξ for $\rho_i = 2\cdot 10^{-1}\,M^4$ and $w_i = 0.5$. The box in the first row on the left shows the absolute values of the leading terms η_2 and η_3, which are roughly one order of magnitude larger than the remaining terms η_l's displayed in the plot on the right along with R_η (the dotted line). In the second row of Fig. 1, the modula of the leading terms ξ_2, ξ_3 and ξ_5 are plotted on the left while the remaining coefficients ξ_l and $R_\xi(y)$ are presented on the right. Thanks to the relatively large amplitudes of the leading terms, one can rely on the second order approximation even when the ρ_i's are not too small, implying that a great improvement over first order results can be achieved in a regime where $\rho_i \sim 10^{-2}$. In this case the errors produced by truncating the expansion to $\mathcal{O}(\epsilon^2)$ are much smaller than one percent.

Let us further consider the particular case $\rho_p = 0$. This choice is made in order to study the cosmological consequences experienced on the negative tension brane (at $y = 1/2$) generated by second order terms proportional to ρ_n^2. We therefore use the proper time τ on the negative tension brane [16]. In each graph in Fig. 2 we plot the squared Hubble parameter on the negative tension brane of Eq. (57) and the acceleration of Eq. (60) for a given energy density ρ_n as a function of its equation of

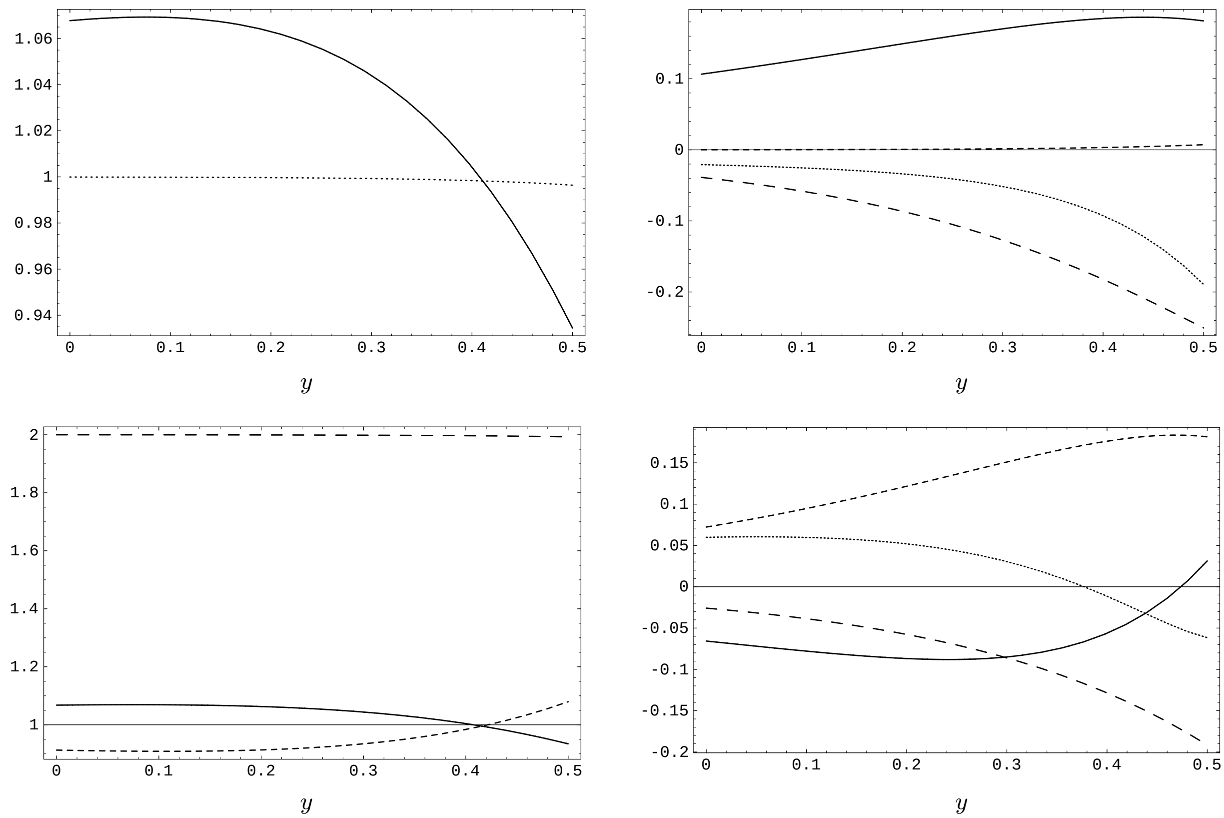

FIG. 1: The graphs on the left contain the plot of the absolute values of the leading terms among η_l (above) and ξ_l (below) to $\mathcal{O}(\epsilon^2)$. The graphs on the right show the subleading terms among η_l (above) and ξ_l (below) and the corresponding remainders R_η and R_ξ to $\mathcal{O}(\epsilon^2)$ (dotted lines). All plots are for $\rho_i = 2 \cdot 10^{-1}\, M^4$ and $w_i = 0.5$ and cover all the bulk between the two branes.

state w_n. The plot on the left in the first row is for the small density $\rho_n = 10^{-2}\, M^4 \ll M^4$ and shows a behavior which is typical of standard four-dimensional cosmology. In fact, only $w_n < -1/3$ leads to an accelerating phase. This trend is modified by higher densities, as it emerges in the remaining graphs of Fig. 2. In particular when $\rho_n = 2 \cdot 10^{-1}\, M^4$ (plot on the left in the second row) the second order corrections seem to provide an accelerated regime for $0 < w_n < 1$. The acceleration appears amplified when $\rho_n = 6 \cdot 10^{-1}\, M^4 < M^4$ (plot on the right in the second row) or higher. Note however that the Hubble parameter, a positive definite quantity, constrains the region swept by w_n which is not allowed to reach unity. Finally the intermediate case is showed in the plot on the right the first row: due to the second order effect, H^2 exhibits a dependence on w_n otherwise not present.

The four regimes described above have to be tested with particular accuracy because the acceleration and Hubble parameter plotted in Fig. 2 are not the leading terms in Eqs. (13) and could thus be comparable with the remainders. In this case, it is somehow possible that the remainders significantly modify the behavior. We first note that $H^2 \sim \eta_1$ and $\ddot{a}/a \sim \xi_1$, as defined above in Eqs. (67a) and (69a), hence we can use η_1 and ξ_1 in place of H^2 and $\ddot{a}/a$ respectively. We then plot in Fig. 3 the ratio between the remainder R_η (evaluated to second order) and the squared Hubble parameter η_1 evaluated to first and second orders. The four plots show this ratio for different choices of ρ_n and w_n (the same as in Fig. 2 and in the same order) as a function of y. In particular, we choose $w_n = 0.95$ in order to explore regions where the acceleration has an unconventional behavior for $\rho_n = 10^{-2}\, M^4$, $10^{-1}\, M^4$, $2 \cdot 10^{-1}\, M^4$ and $w_n = 0.65$ for $\rho_n = 6 \cdot 10^{-1}\, M^4$. Apart from the last case, the corrections given by the neglected terms cannot significantly modify H^2 on the negative tension brane. In the first line of Fig. 4, the ratios between R_ξ (to second order) and the acceleration ξ_1 to first and second order are analogously plotted. For low enough energy densities, the second order expressions appear to be good approximations. On the other hand, in the second row it is shown that, in the unconventional regime of the acceleration (that is $\rho_n = 2 \cdot 10^{-1}\, M^4$ and $w_n = 0.95$, and $\rho_n = 6 \cdot 10^{-1}\, M^4$ and $w_n = 0.65$) the two terms are of the same order of magnitude. This shows that one should go beyond $\mathcal{O}(\epsilon^2)$ in order to determine the true behavior of $\ddot{a}/a$ for such equations of state.

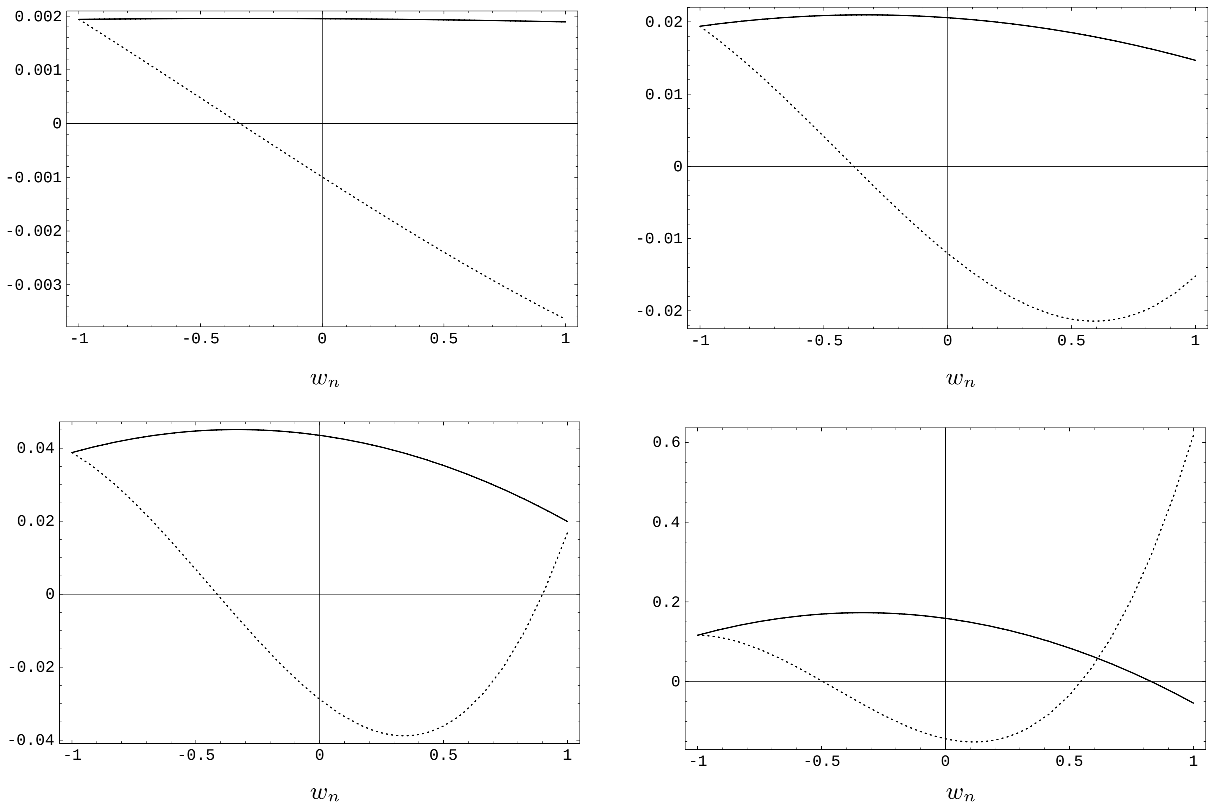

FIG. 2: Plots of $H^2(1/2,\tau)$ (solid line) and $\ddot{a}(1/2,\tau)/a(1/2,\tau)$ (dotted line) to $\mathcal{O}(\epsilon^2)$ at a given time, as functions of w_n, for $\rho_n = 10^{-2}\,M^4$, $10^{-1}\,M^4$, $2\cdot 10^{-1}\,M^4$ and $6\cdot 10^{-1}\,M^4$ (from top left to bottom right).

VI. CONCLUSIONS

We have computed approximate cosmological solutions of five-dimensional Einstein equations for Randall-Sundrum models in the presence of a radion effective potential. The calculations were performed up to the second order in the energy densities of the matter on the branes and assuming an adiabatic evolution of the system. Our approach differs from Ref. [10] in that we do not include a specific bulk field to achieve stabilization, and is therefore more general. Interestingly, their results are recovered in the limit of very large warp factor and radion mass. For the RS I model with matter localized only on the negative tension brane, we found negligible corrections for the Hubble parameter in the case of radiation or cosmological constant, thus supporting one of the main results of Ref. [10]. For RS II, we found negligible corrections for the equations of state just described and in the limit when the distance between the branes is taken to infinity.

On inspecting our results, we finally found some evidence of an accelerating phase for a wider range of values of the equation of state $p_n = w_n\,\rho_n$ on the negative tension brane if the distance between the branes is finite. However, one should then carry the computation to higher orders, since such an effect appears near the limit of validity of our perturbative expansion.

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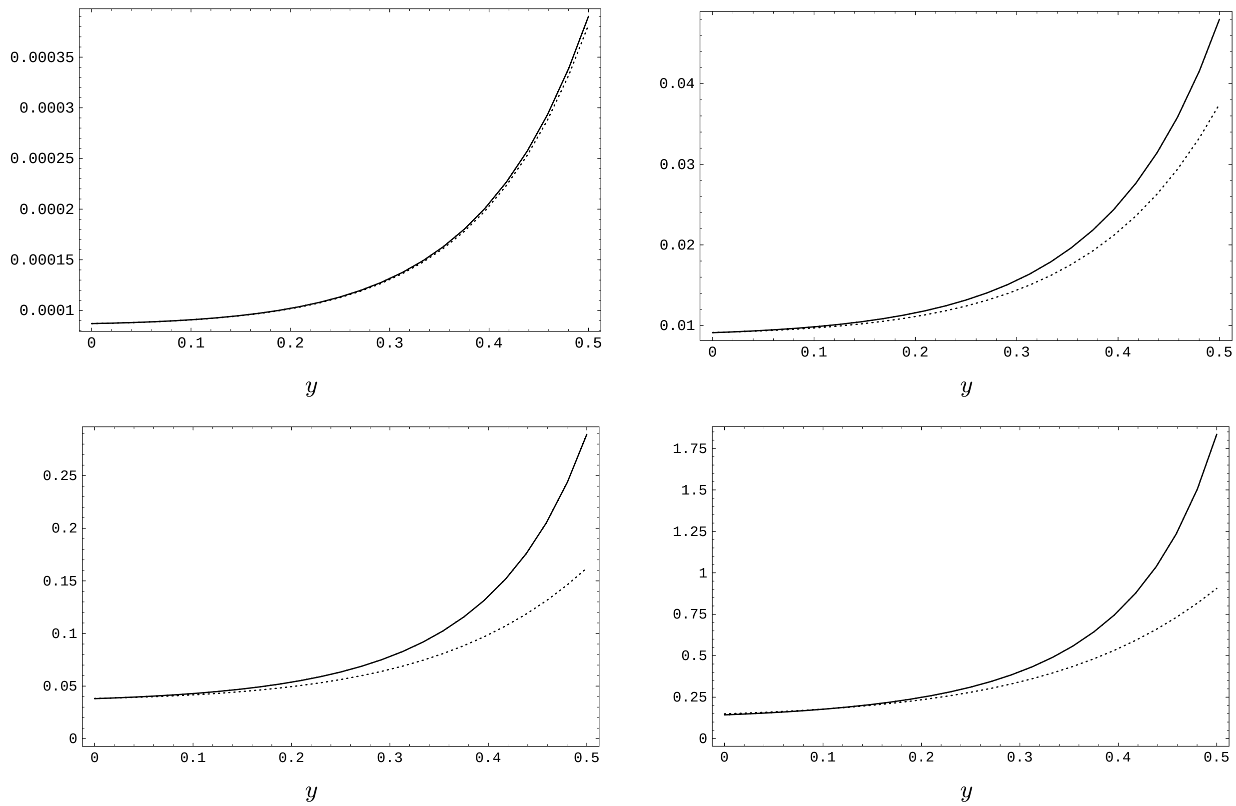

FIG. 3: The two graphs in the first row show the ratio between R_η to $\mathcal{O}(\epsilon^2)$ and $\eta_1 \sim H^2$ to $\mathcal{O}(\epsilon)$ (dotted line) and to $\mathcal{O}(\epsilon^2)$ (solid line) at a given time, for $\rho_n = 10^{-2}\,M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 10^{-1}\,M^4$ and $w_n = 0.95$ (right). In the two graphs in the second row, the same ratios are are given for $\rho_n = 2 \cdot 10^{-1}\,M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 6 \cdot 10^{-1}\,M^4$ and $w_n = 0.65$ (right). The plots cover all the bulk between the two branes.

[12] Note that the approximation which makes use of an effective Lagrangian is only compatible with second order calculations.

[13] Note that this conservation equation can be obtained by taking the limit $y \to y_i$ in the equation $G_{04} = 0$.

[14] The unperturbed brane distance is taken to be finite and equal to $b_0/2$.

[15] In terms of the fundamental scale M one has $m = M$, $k^2 = M^{-3}$, $b_0 = M^{-1}$, and $\omega^2 = M^7$.

[16] Let us recall that this is achieved by adopting the particular gauge choice for $c^{(1)}_{n,i}$ and $c^{(2)}_{n,i}$ which sets $f^{(1)}_{n,i}(t) = f^{(2)}_{n,i}(t) = 0$ and $c^{(1)}_{a,i} = c^{(2)}_{a,i} = 0$.

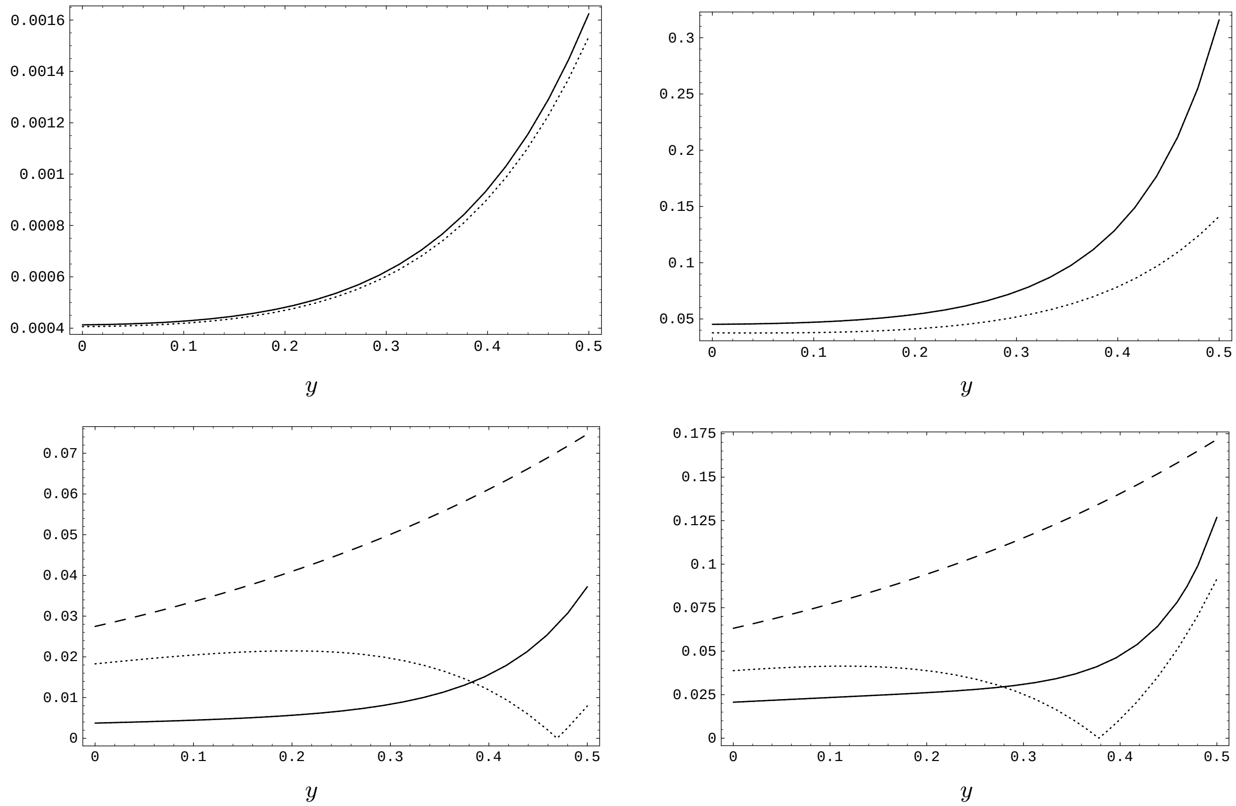

FIG. 4: The two graphs in the first row show the ratio between R_ξ to $\mathcal{O}(\epsilon^2)$ and $\xi_1 \sim \ddot{a}/a$ to $\mathcal{O}(\epsilon)$ (dotted line) and to $\mathcal{O}(\epsilon^2)$ (solid line) at a given time, for $\rho_n = 10^{-2}\,M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 10^{-1}\,M^4$ and $w_n = 0.95$ (right). In the two graphs in the second row, R_ξ to $\mathcal{O}(\epsilon^2)$ (solid line) is compared to the modula of ξ_1 to $\mathcal{O}(\epsilon)$ (dashed line) and to $\mathcal{O}(\epsilon^2)$ (dotted line) for $\rho_n = 2 \cdot 10^{-1}\,M^4$ and $w_n = 0.95$ (left), and for $\rho_n = 6 \cdot 10^{-1}\,M^4$ and $w_n = 0.65$ (right). The plots cover all the bulk between the two branes.